Deciding Fuzzy Description Logics by Type Elimination

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Motivation

Fuzzy Description Logics (FDLs):

- DLs that can represent vague concepts and relations
- Extend classical (crisp) DLs semantically and syntactically

Complexity of Reasoning in FDLs?

- We could not find too many results, expect e.g. [Str01]
- **Intuitively:** extension of classical DLs implies “at least as hard as DLs”
- It is good to have a variety of different tools to tackle the problem!

Reasoning in FDLs

- **Directly** in the FDL world: mainly tableau-based methods, e.g. [Str01, SSSP06, SSP$^+$07, SB07, LXLK06, HPS08]
- **By translation** to classical DLs, e.g. [Str04] (should always lead to a suboptimal solution!)

Research Question

Can we come up with a **new way** to perform reasoning in FDLs that

- Solves a fundamental and important reasoning problem
- Works directly at the level of FDLs (no translation to DLs)
- Works differently from tableau-based methods
- Can deal with general (unrestricted) terminologies, i.e. GCIs?

What did we achieve?

- Designed a novel procedure $\mathbf{FixIt}(\mathbb{ALC})$ for deciding knowledge base (KB) satisfiability in the FDL $\mathbb{ALC}$
- Formally proved soundness, completeness and termination of the algorithm and can show that the runtime behavior is worst-case optimal
- It is the first fixpoint-based decision procedure that has been proposed for FDL introducing a new class of inference procedures into FDL reasoning
- Our approach can deal with general terminologies (GCIs)
  - Together with [SSSP06, LXLK06, SB07, HPS08], one of the few possible approaches.
  - First non-tableau-based decision procedure to integrate GCIs
  - General terminologies are handled differently than in standard tableau-based method such as [SSSP06, LXLK06]

How did we achieve that?

- **FixIt**($\mathbb{ALC}$) generalizes a type-elimination-based decision procedure [Pra80] for the (classical) modal logic $\mathbf{K}$, i.e. $\mathcal{KBDD}$ [PSV06], to the FDL $\mathbb{ALC}$
- Principle underlying $\mathcal{KBDD}$ carries over to $\mathbb{ALC}$, but only in a different form than in [PSV06]
- Additionally, we integrate (fuzzy) ABoxes and (general) TBoxes which are not dealt with in $\mathcal{KBDD}$

A (Minimalist) Fuzzy Description Logic: $\mathbb{ALC}$ [Str01]

Syntax and Semantics of Concept Expressions

Constructor	**Syntax** E	**Semantics** $E^{\mathcal{I}}(o)$ (wrt. interp. $\mathcal{I}$)
concept names	A	$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1]$
role names	R	$R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1]$
universal truth / possibility	$\top$	1
concept conjunction	$C \sqcap D$	$min(C^{\mathcal{I}}(o), D^{\mathcal{I}}(o))$
concept disjunction	$C \sqcup D$	$max(C^{\mathcal{I}}(o), D^{\mathcal{I}}(o))$
concept negation	$\neg C$	$1 - C^{\mathcal{I}}(o)$
univ. value restriction	$\forall R.C$	$inf_{o' \in \Delta^{\mathcal{I}}}\{max(1 - R^{\mathcal{I}}(o,o'), C^{\mathcal{I}}(o'))\}$
ex. value restriction	$\exists R.C$	$sup_{o' \in \Delta^{\mathcal{I}}}\{min(R^{\mathcal{I}}(o,o'), C^{\mathcal{I}}(o'))\}$
universal falsehood	$\bot$	0

A (Minimalist) Fuzzy Description Logic: $\mathbb{ALC}$ [Str01]

Syntax and Semantics of Fuzzy Axioms

Type	Axiom α	Satisfaction of α by $\mathcal{I}$: $\mathcal{I} \models \alpha$
$\mathcal{T}$: General Concept Inclusion (GCI)	$C \sqsubseteq D$	$C^{\mathcal{I}}(o) \leq D^{\mathcal{I}}(o)$ for all $o \in \Delta^{\mathcal{I}}$
$\mathcal{A}$: Fuzzy concept membership	$\langle i : C \bowtie d \rangle$	$C^{\mathcal{I}}(i^{\mathcal{I}}) \bowtie d$ $\bowtie \in \{\geq, \leq, =\}$
$\mathcal{A}$: Fuzzy relation assertion	$\langle R(i, i') \geq d \rangle$	$R^{\mathcal{I}}(i^{\mathcal{I}}, i'^{\mathcal{I}}) \geq d$

Syntax and Semantics of a fuzzy KB

A **TBox** $\mathcal{T}$ is a finite set of GCIs. An **ABox** $\mathcal{A}$ is a finite set of fuzzy assertions. A **fuzzy knowledge base** $\mathcal{K}$ is a pair $\mathcal{K} = (\mathcal{T}, \mathcal{A})$

$\mathcal{I} \models \mathcal{T}$ iff. $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{T}$

$\mathcal{I} \models \mathcal{A}$ iff. $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{A}$

$\mathcal{I} \models \mathcal{K}$ iff. $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$

Reasoning in $\mathbb{ALC}$

Given a fuzzy KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, fuzzy ABox axioms or GCIs α, we can analyze particular semantic characteristics and interdependencies:

- $\mathcal{K}$ is **satisfiable** (or consistent) iff there is a model $\mathcal{I}$ for $\mathcal{K}$, i.e. an interpretation $\mathcal{I}$ such that

$$\mathcal{I} \models \mathcal{T} \quad \text{and} \quad \mathcal{I} \models \mathcal{A}$$

- $\mathcal{K}$ **entails** α (denoted as $\mathcal{K} \models \alpha$) iff all models $\mathcal{I}$ of $\mathcal{K}$ satisfy α

Specific entailment problems: concept equivalence, subsumption, disjointness, concept membership at least / most to a given degree, individual interrelation at least to a given degree, ...

Complexity of checking KB satisfiability in $\mathbb{ALC}$:

ExpTime-complete [KH09, KH08]

Complexity of Decision Problems in $\mathbb{ALC}$

Results from [KH09, KH08]:

Reasoning Task	**Worst-case Complexity**
KB satisfiability	EXPTIME-complete
Terminological Reasoning	
$\mathcal{K} \models C \equiv D$	EXPTIME-complete
$\mathcal{K} \models C \sqsubseteq D$	EXPTIME-complete
Concept Disjointness wrt. $\mathcal{K}$	EXPTIME-complete
Concept Satisfiability wrt. $\mathcal{K}$	EXPTIME-complete
Extensional Reasoning	
$\mathcal{K} \models \langle o : C \geq n \rangle$	EXPTIME-complete
$\mathcal{K} \models \langle o : C \leq n \rangle$	EXPTIME-complete
$\mathcal{K} \models \langle o : C = n \rangle$	EXPTIME-complete
$\mathcal{K} \models \langle R(o, o') \geq n \rangle$	EXPTIME-hard

Type Elimination: Overview

- Can be seen as a **model building procedure**
- Does not rely on systematic search in the first place, but instead constructs a canonical interpretation by means of a **fixpoint construction**
- The computed interpretation is in general **not tree-shaped** (as in tableau-based methods)

Closure of a Knowledge Base

Intuitively: the closure of a KB is a set of concept that contains for any concept expression occurring in KB the positive and negative form

Example:
$\mathcal{K} = (\{Human \sqsubseteq \neg\exists hasParent.\neg Human\}, \{\langle john : Human \sqcap Saint \geq 0.8\rangle\})$

$$\begin{array}{rl}\mathsf{cl}(\mathcal{K}) = \{ & Human, \neg Human, \\ & \exists hasParent.\neg Human, \neg\exists hasParent.\neg Human, \\ & Human \sqcap Saint, \neg(Human \sqcap Saint), \\ & Saint, \neg Saint \quad \}\end{array}$$

- We can state for any relevant (complex) property C that some individual i in $\mathcal{K}$ could have, if the the property applies (C), or does not apply ($\neg C$)

How we use that: The closure of a KB gives us the basic vocabulary to describe all relevant properties of individuals in interpretations

Relevant Possibility Degrees

Further, let PossDeg$(\mathcal{K})$ denote the set of all **relevant possibility degrees** that can be derived from $\mathcal{K}$ defined by

$$\text{PossDeg}(\mathcal{K}) = \{0, 0.5, 1\} \cup \{d \mid \langle \alpha \geq d \rangle \in \mathcal{A}\} \cup \{1 - d \mid \langle \alpha \geq d \rangle \in \mathcal{A}\}$$

[Str01, Str04] showed that

$\mathcal{K}$ is satisfiable
iff
there a model of $\mathcal{K}$ which assigns possibility degrees in PossDeg$(\mathcal{K})$ only

Hence: we do not need to consider arbitrary possibility degrees $d \in [0, 1]$, but only the *finite* set PossDeg$(\mathcal{K})$ that can be derived from $\mathcal{K}$.

Fuzzy Types

We are living in a fuzzy world: Properties always hold to a certain degree

Definition (Fuzzy $\mathcal{K}$-Type)

A *fuzzy $\mathcal{K}$-type* τ is a maximal subset of cl$(\mathcal{K}) \times$ PossDeg$(\mathcal{K})$ such that the following conditions are satisfied:

1. if $\langle C, d\rangle \in \tau$ and $\langle C, d'\rangle \in \tau$ then $d = d'$
2. if $C = \neg C'$ then $\langle C, d\rangle \in \tau$ iff $\langle C', 1 - d\rangle \in \tau$
3. if $C = C' \sqcap C''$ then $\langle C, d\rangle \in \tau$ iff $\langle C', d'\rangle \in \tau$ and $\langle C'', d''\rangle \in \tau$ and $d = min(d', d'')$
4. if $C = C' \sqcup C''$ then $\langle C, d\rangle \in \tau$ iff $\langle C', d'\rangle \in \tau$ and $\langle C'', d''\rangle \in \tau$ and $d = max(d', d'')$
5. for all $C \sqsubseteq C' \in \mathcal{T}$: if $\langle C, d\rangle \in \tau$ and $\langle C', d'\rangle \in \tau$ then $d \leq d'$
6. if $C = \top$ then $\langle C, 1\rangle \in \tau$.

Fuzzy Types vs. Individuals

Individuals

- Basic elements to compose interpretations
- Are assigned elementary properties that can be observed (wrt. an interpretation)
- Are interrelated with other individuals (to a certain degree) in an $\mathcal{I}$

Fuzzy Types:

- Syntactic view on the properties of a *possible* individual i
- State what elementary *and* complex properties can be observed about i
- Consistent with the semantics of the boolean constructors
- Modal properties (e.g. $\forall R.C$) constrain the way i can be interrelated to other individuals i' in an $\mathcal{I}$

Hence: Types are syntactic correspondents to individuals (used in interpretations for $\mathcal{K}$)

$\langle C, d\rangle \in \tau \sim$ the individual represented by τ is a member of C to degree d

Canonical Model

Set of all Types = Vocabulary to construct any interpretation $\mathcal{I}$ of $\mathcal{K}$

- We simply need to fix how to interconnect the individuals they represent

Canonical Interconnection and Interpretation:

- Given a set T of $\mathcal{K}$-types, interconnect them in a standard (or *canonical* way) $\Delta_R(\tau, \tau')$

  $$\Delta_R(\tau, \tau') := min\{\delta(d, d') | \langle \forall R.C, d\rangle \in \tau, \langle C, d'\rangle \in \tau'\}$$

  with $\delta(d, d') := 1$ if $d \leq d'$ and $\delta(d, d') := 1 - d$ if $d > d'$
- The resulting canonical interpretation $\mathcal{I}(T)$ is **almost directly** a model of the TBox $\mathcal{T}$ of the input KB $\mathcal{K}$

$$C^{\mathcal{I}(T)}(\tau) = d \text{ iff } \langle C, d\rangle \in \tau \qquad (*)$$

for *almost* all $C \in \text{cl}(\mathcal{K})$.

If $(*)$ would be satisfied for all $C \in \text{cl}(\mathcal{K})$, then we would have $\mathcal{I}(T) \models C \sqsubseteq C'$ for all $C \sqsubseteq C' \in \mathcal{T}$ by clause (5) in our definition of $\mathcal{K}$-types, i.e. our canonical interpretation would be a model for $\mathcal{T}$.

What is the Problem with the Canonical Interpretation?

We know that

$$C^{\mathcal{I}(T)}(\tau) = d \text{ iff } \langle C, d \rangle \in \tau \qquad (*)$$

for *almost* all $C \in \mathsf{cl}(\mathcal{K})$.

That $(*)$ is satisfied by $\mathcal{I}(T)$ is straightforward for the cases of concept names C, $\top$, or complex concepts of the form $C = C' \sqcap C''$, $C = C' \sqcup C''$, $C = \neg C'$, as well as the $C^{\mathcal{I}(T)}(\tau) \geq d$ case for $C = \forall R.C$ by our definition of types and the definition of Δ_R.

The only cases where $(*)$ can be violated by $\mathcal{I}(T)$ is for types τ containing universally role restricted concepts $\forall R.C$ that are assigned a possibility degree which is *too small* (wrt. the R-successor types τ' in $\mathcal{I}(T)$) to properly reflect the semantics of $\forall R.C$ in $\mathbb{ALC}$, i.e. to coincide with the *greatest* lower bound of the set

$$\{max(1 - R^{\mathcal{I}(T)}(\tau, \tau'), C^{\mathcal{I}(T)}(\tau')) \mid \tau' \in T\}$$

How to fix this Problem?

Call any type τ containing universally role restricted concepts $\forall R.C$ that are assigned a possibility degree which is *too small* (wrt. the R-successor types τ' in $\mathcal{I}(T)$) to properly reflect the semantics of $\forall R.C$ in $\mathbb{ALC}$, i.e. does not coincide with the *greatest* lower bound of the set

$$\{max(1 - R^{\mathcal{I}(T)}(\tau, \tau'), C^{\mathcal{I}(T)}(\tau')) \mid \tau' \in T\}$$

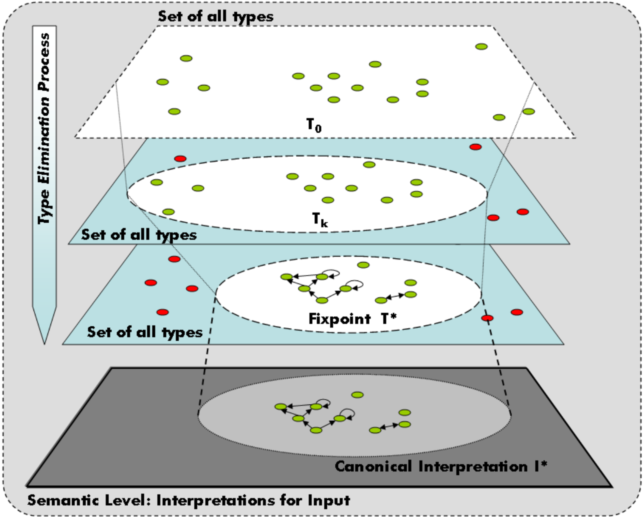
a **bad type** (wrt. the given set of types T).

Bad types $\tau \in T$ can be detected easily: badtype(τ) iff

$\langle \forall R.C, d \rangle \in \tau$ and for all $\tau' \in T$: if $\langle C, d' \rangle \in \tau'$ then $max(1 - \Delta_R(\tau, \tau'), d') > d$

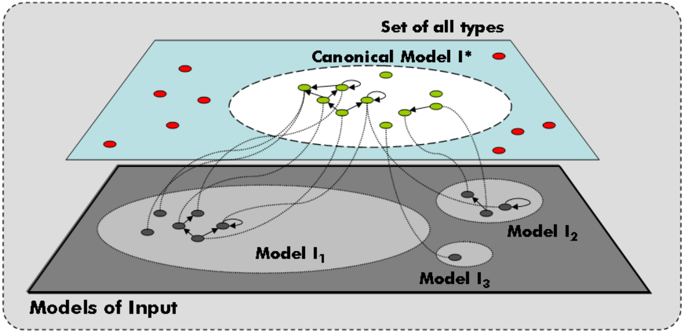
Basic Idea: Bad types are the only trouble that prevent us from ending up with a model for $\mathcal{T}$. $\rightarrow$ **Remove (iteratively) all bad types from T!**

Computation of a Canonical Model by Type Elimination



- For the fixpoint set T^* the canonical interpretation $\mathcal{I}(T^*)$ is a model of $\mathcal{T}$

The Canonical Model is the “Maximal Model”



Lemma

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be any model of $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. For each individual $o \in \Delta^{\mathcal{I}}$ we define its corresponding type

$$\tau(o) := \{\langle C, d\rangle \mid C^{\mathcal{I}}(o) = d\}$$

Then, $\Delta_R(\tau(o), \tau(o')) \geq R^{\mathcal{I}}(o, o')$ for all $o, o' \in \Delta^{\mathcal{I}}$.

- **Last step:** Check an ABox model can be derived from $\mathcal{I}(T^*)$

Our Decision Procedure **FixIt**$(\mathbb{ALC})$

procedure satisfiable($\mathcal{K}$): boolean
T := $\{\tau | \tau$ is a $\mathcal{K}$-type $\}$;
repeat
  $T' := T$;
  $T := T' \setminus badtypes(T')$;
until $T = T'$;
if *there exists a total function* $\pi : Ind_{\mathcal{A}} \to T$ *s.t.* $\langle C, d' \rangle \in \pi(o)$ *and* $d \leq d'$ *for each* $\langle o : C \geq d \rangle \in \mathcal{A}$, *and* $\Delta_R(\pi(o), \pi(o')) \geq d$ *for each* $\langle R(o, o') \geq d \rangle \in \mathcal{A}$ **then**
  return true;
end
return false;

function $badtypes(T) : 2^T$
return $\{\tau \in T | \langle \forall R.C, d \rangle \in \tau$ and for all $\tau' \in T$: if $\langle C, d' \rangle \in \tau'$ then $max(1 - \Delta_R(\tau, \tau'), d') > d\}$;

Future Work

- Study means of implicit representation of sets of fuzzy types known from Symbolic Model Checking [McM93], in particular OBDDs
- A major question concerning optimization: how to implement the final test of the algorithm efficiently, e.g. by heuristic search using the information in the ABox effectively to find the required mapping
- The integration of optimizations such as full vs. lean representations or particle vs. types as discussed in [PSV06]
- Evaluate the efficiency of the method by implementation and comparison to tableau-based systems for FDLs
- Study the bottom-up variant of $\mathcal{KBDD}$ in the context of FDLs too, check if the integration of ABoxes can be done more efficiently in such a variant.
- Investigate to what extend the method can cover other semantics for FDLs (e.g. other t-norms) and extended constructs, such as fuzzy modifiers and concrete domains

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